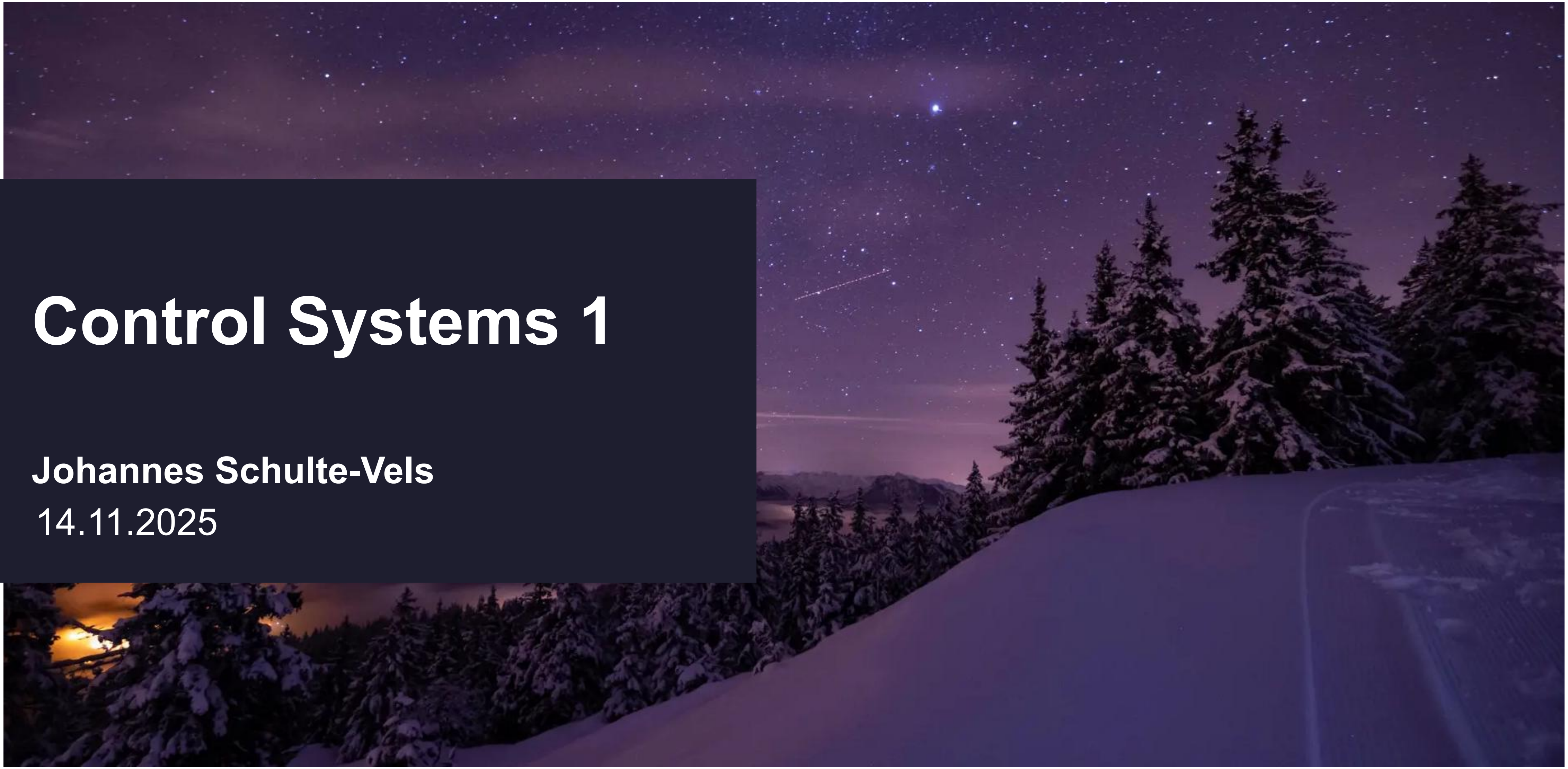
ETH zürich

Institute for Dynamic Systems and Control



Control Systems 1

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Welcome!

Polybox

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Today

- Repetition Session 8 (brief)
- Theory Recap
  - Frequency Domain
  - Bode Plot
- Q&A Session / Done

Repetition Session 8

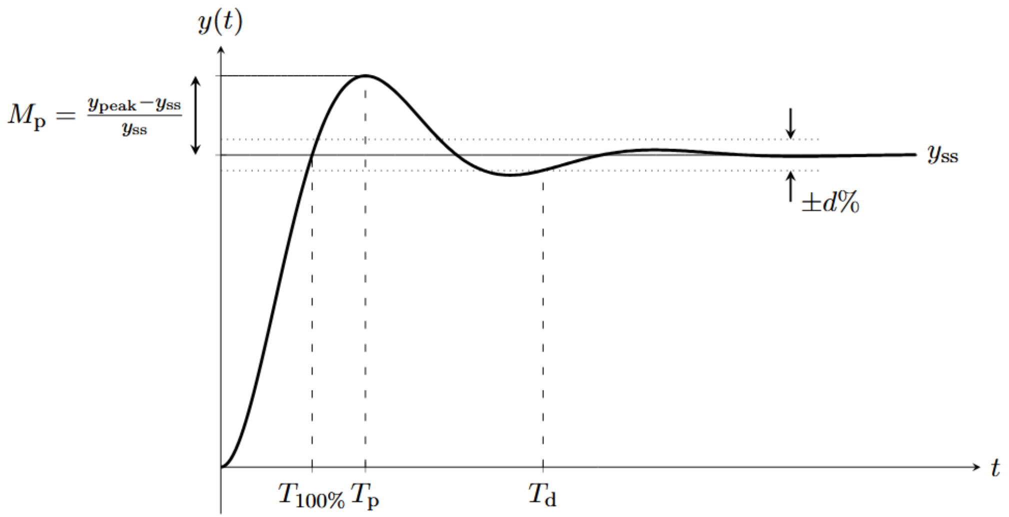
Time-Domain Specifications

Often times, we do not just want our system to be stable, we want it to have certain features.

Example: When aiming for a certain velocity in a car, we do not only want to be sure that we don't accelerate infinitely, but we also care about how fast, how smooth and how initially precise we can reach our target velocity.

For this purpose we introduced **Time-Domain Specifications**, that help us characterise the transient response and set certain constraints. This is mainly done by **controlling the position of the poles** in our system!

Time-Domain Specifications



- T_d, **Settling Time**: Time for step response to be withing d% of steady state response

$$T_d = \frac{1}{|\sigma|} \ln\left(\frac{100}{d}\right)$$

- T_p, **Time to Peak**: Time for step response to reach highest value

$$T_p = \frac{\pi}{\omega}$$

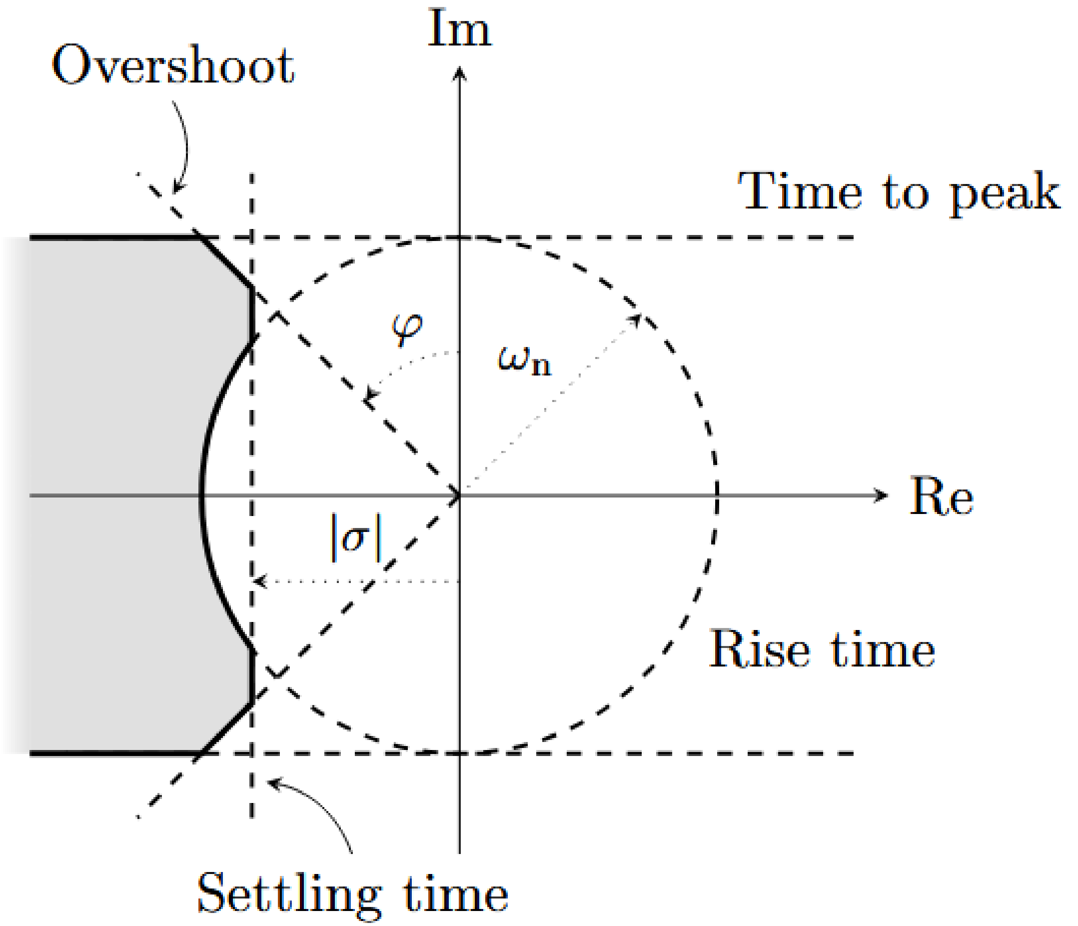
- M_p, **Peak Overshoot**: Ration beween highest value and steady state value (gain)

$$M_p = e^{\frac{\sigma \pi}{\omega}}$$

- $T_{100\%}$, **Rise Time**: Time for step response to reach the steady state gain (for the first time)

$$T_{100\%} = \frac{\frac{\pi}{2} - \varphi}{\omega} \approx \frac{\pi}{2\omega_n}$$

Time Domain Specifications



Dominant Pole Approximation

We want to be able to analyse the systems behaviour and set Time-Domain Specifications. However, we only looked at how to do this for 1. and 2. order systems. When going in higher orders, this gets way more complicated.

For that we introduced **dominant pole approximation**, basically saying that the systems behaviour can be approximated by only looking at the dominant (**limiting**) poles.

Remember how for the impulse response, the **real part of every pole determines how fast / how slow the exponential decreases**. The more negative the real part, the faster it decreases.

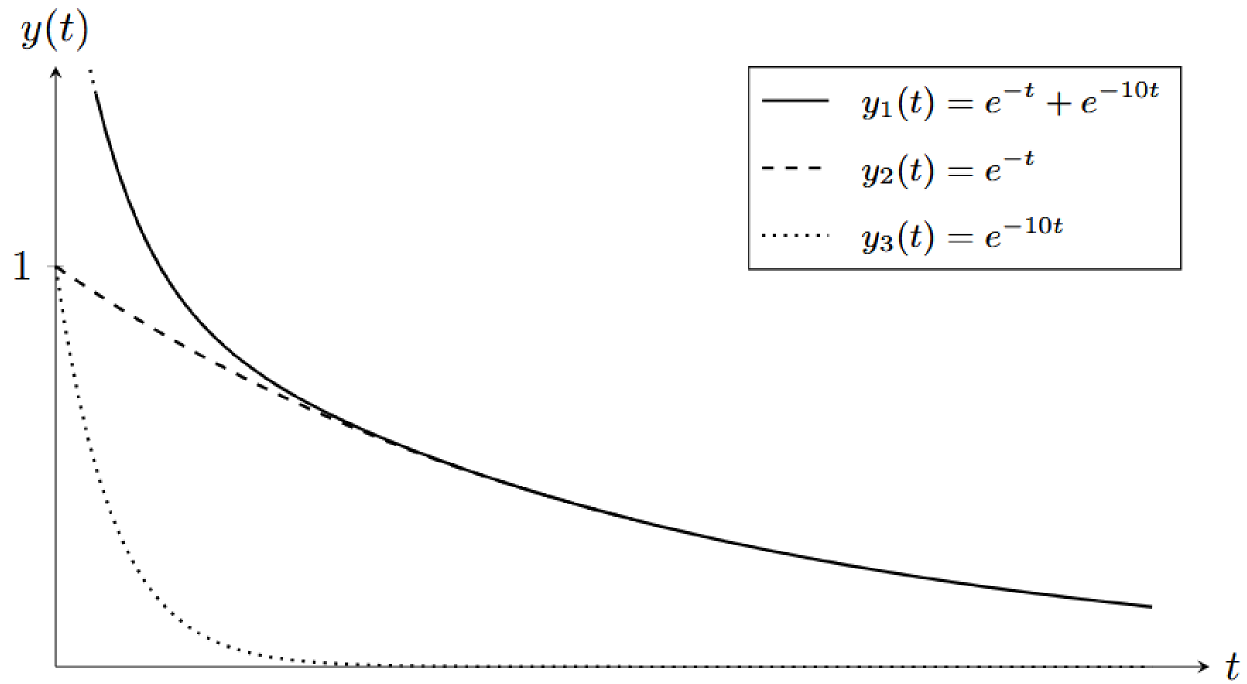
$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}$$

$$\mathcal{L}^{-1}\left\{\frac{r_i}{s-p_i}\right\} = r_i e^{p_i t}$$

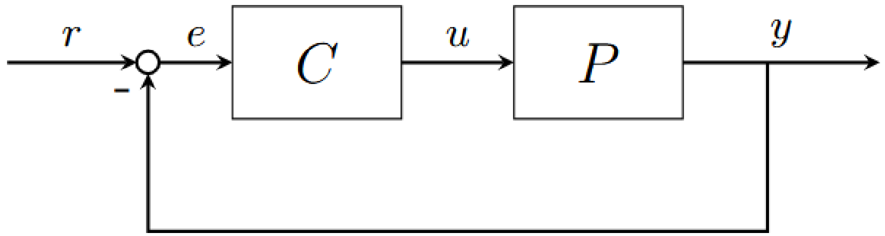
$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}, \quad t \geq 0$$

Dominant Pole Approximation

Below we can see that the response of the system can be well approximated by the slower, limiting poles. By only looking at them, we can **«reduce» the order of our system**!



PID Control Motivation



$$e(t) = r(t) - y(t)$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

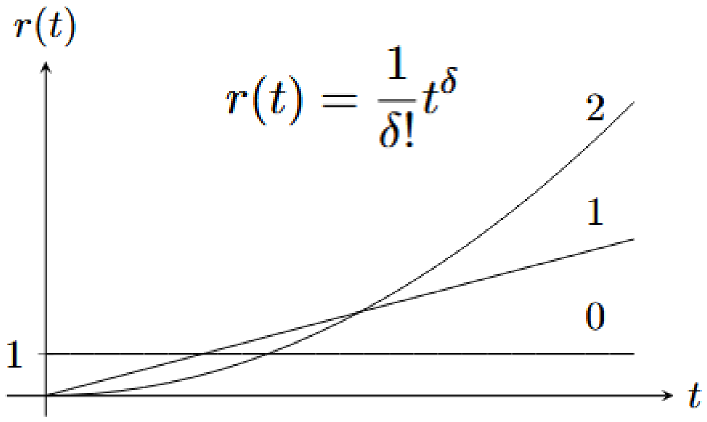
Ideally, we would like the error to be zero for all times, such that the the output is always equal to the reference.
But we already know this is not possible, as the system needs time to adapt to a new reference (remember **transient** and **steady state response**).

We can however look at the **steady state error e_{ss}** , that allows us to asses what would be necessary for the system to reach zero e_{ss} .

Steady State Error (SSE)

The table to the right gives us insight about what SSE is present for what number of integrators and what degree of input. The formula above that gives a more mathematical insight.

Remember that q is the number of integrators present in the system, and the δ (=delta) arises from the type of reference we input (below)



$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + \frac{k_{Bode}}{s^q}} \cdot \frac{1}{s^{\delta}}$$

e_{ss}	$\delta = 0$	$\delta = 1$	$\delta = 2$
$q = 0$	$\frac{1}{1+k_{Bode}}$	∞	∞
$q = 1$	0	$\frac{1}{k_{Bode}}$	∞
$q = 2$	0	0	$\frac{1}{k_{Bode}}$

PID Controller

A (PID) controller creates an input signal to the plant that depends on the negative error e, the integral of e and the derivative of e:

$$u(t) = k_p \cdot e(t) + k_i \cdot \int_0^t e(\tau)d\tau + k_d \cdot \frac{de(t)}{dt}$$

The transfer function of a PID controller is:

$$C_{\text{PID}}(s) = k_d s + k_p + \frac{k_i}{s} = \frac{k_d s^2 + k_p s + k_i}{s}$$

	Advantages	Disadvantages
k_p	e_{ss} decreases, Faster response, Increases Bandwidth	More sensitive to noise, More oscillations, Phase margin decreases
k_d	Less oscillations, Phase margin increases, Reduces overshoot	More sensitive to noise, Slower response
k_i	e_{ss} decreases, Faster response	More oscillations, Phase margin decreases

Initial and Final Value Theorem

For you to look at if you want. Won't go through it now. However it is nice! (These 4 Slides)

We often want to know what the steady state output of the system is. But being in the s-domain doesn't make it very interpretable. Therefore we introduce a theorem to **link the transfer function to the steady state output (s-domain to t-domain).**

Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Initial Value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to \infty} sF(s)$$

Application Final Value Theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)U(s)$$

Final Value Theorem Proof (No black magic)

1. General true expression for Laplace Transform $\mathcal{L}\{f'(t)\} \;=\; sF(s) - f(0).$

2. Rearrange and reformulate $sF(s) \;=\; \mathcal{L}\{f'(t)\} + f(0) \;=\; \int_0^{\infty} e^{-st} f'(t)\,dt \;+\; f(0).$

3. Take limes $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \int_0^{\infty} e^{-st} f'(t)\,dt + f(0) = \int_0^{\infty} \lim_{s\to 0} e^{-st} f'(t)\,dt + f(0) = \int_0^{\infty} f'(t)\,dt + f(0).$

4. $\lim_{s\to 0} sF(s) = \big[f(\infty) - f(0)\big] + f(0) = f(\infty).$

$\int_0^{\infty} f'(t)\,dt = f(\infty) - f(0)$

5. Reformulate

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Example

$$G(s) = \frac{0.2}{s^2 + s + 2}$$

Calculate the steady state respone for G(s) when applying a dirac impulse as an input $u(t) = \delta(t)$

Example Solution

$$G(s) = \frac{0.2}{s^2 + s + 2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Calculate the steady state respone for G(s) when applying a dirac impulse as an input $u(t) = \delta(t)$

Solving for the roots in the denominator we gat as poles: $\frac{-1 \pm j\sqrt{7}}{2}$ $Re(p_i) < 0 \Rightarrow$ Asympt. Stable

Laplace Transform of Dirac Impulse: $U(s) = 1$

In the Laplace Domain and final value theorem:

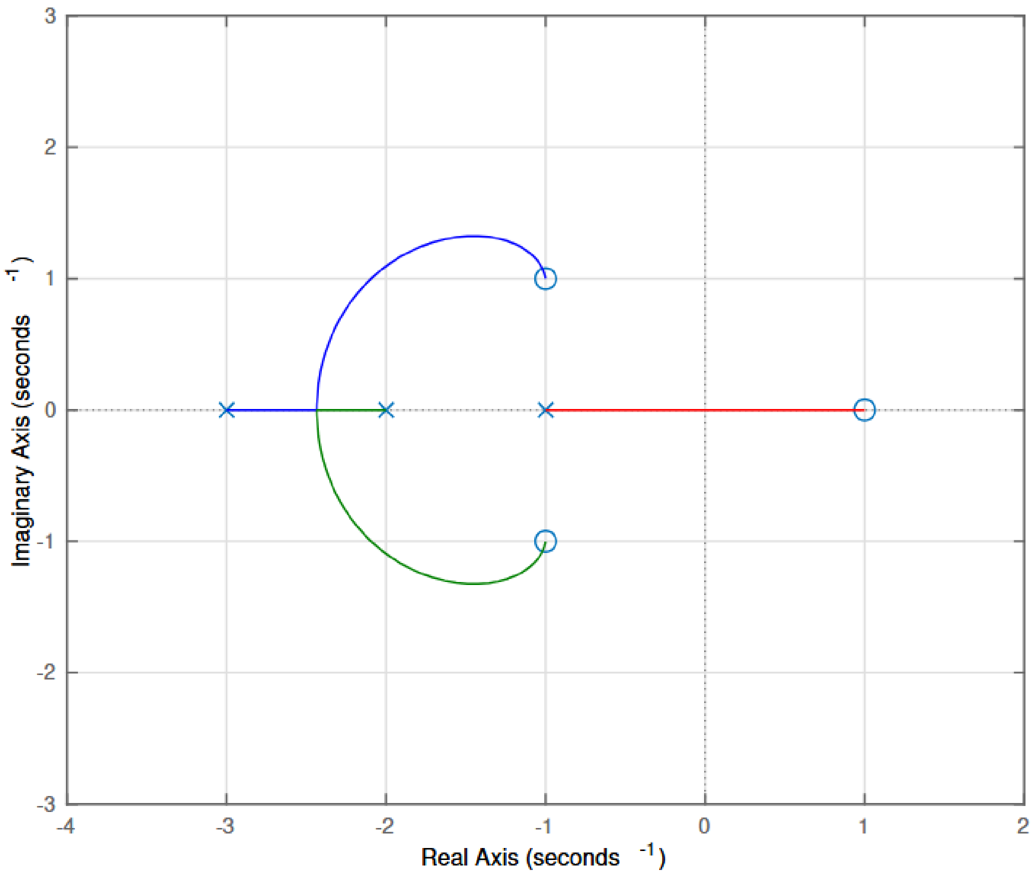
$$Y(s) = G(s)U(s) = G(s)$$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\,Y(s) = \lim_{s \to 0} s\,G(s).$$

Compute the limit and response:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \frac{0.2\,s}{s^2 + s + 2} = 0.$$

FS 2024



Mark the correct statements

1. The open-loop system L is a minimum-phase system.
2. The open-loop system L is asymptotically stable.
3. There exists a gain k^* such that for all k such that $0 < k \leq k^*$ the closed-loop system T is asymptotically stable and all poles of the closed-loop system have zero imaginary part, i.e. are real numbers.

1 2 3

Theory Recap

Frequency Domain

Frequency Response

In the next weeks we are going to look at the **systems response to sinusoidal inputs** and its behaviour. Since a sinusoid is always characterized by it's frequency, we will call this the frequency response.

In contrast to the precvious weeks, this will especially help us characterize the **steady state response**, and not the transient one.

Motivation

(see week 6)

We want to write $u(t) = \cos(\omega t)$, as an function of complex exponential functions:

$$u(t) = \cos(\omega t) = \sum_i U_i e^{s_i t}$$

Now we need to decompose the cosine. Remember how cos can be written:

$$\cos(\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}.$$

So now we see that $U_{1,2} = \frac{1}{2}$ and $s_{1,2} = \pm j\omega.$

Now we want to compute the steady state response $y_{ss}(t) = G(s)e^{st}$ or more generally:

$$y_{ss}(t) = \sum_i U_i G(s_i) e^{s_i t}.$$

Plugging in the values we get: $$y_{ss}(t) = \frac{1}{2}G(j\omega)e^{j\omega t} + \frac{1}{2}G(-j\omega)e^{-j\omega t}.$$

Let's use the fact that $G(j\omega)$ **is a complex function** that can be decomposed into a magnitude and a phase (in polar form) .

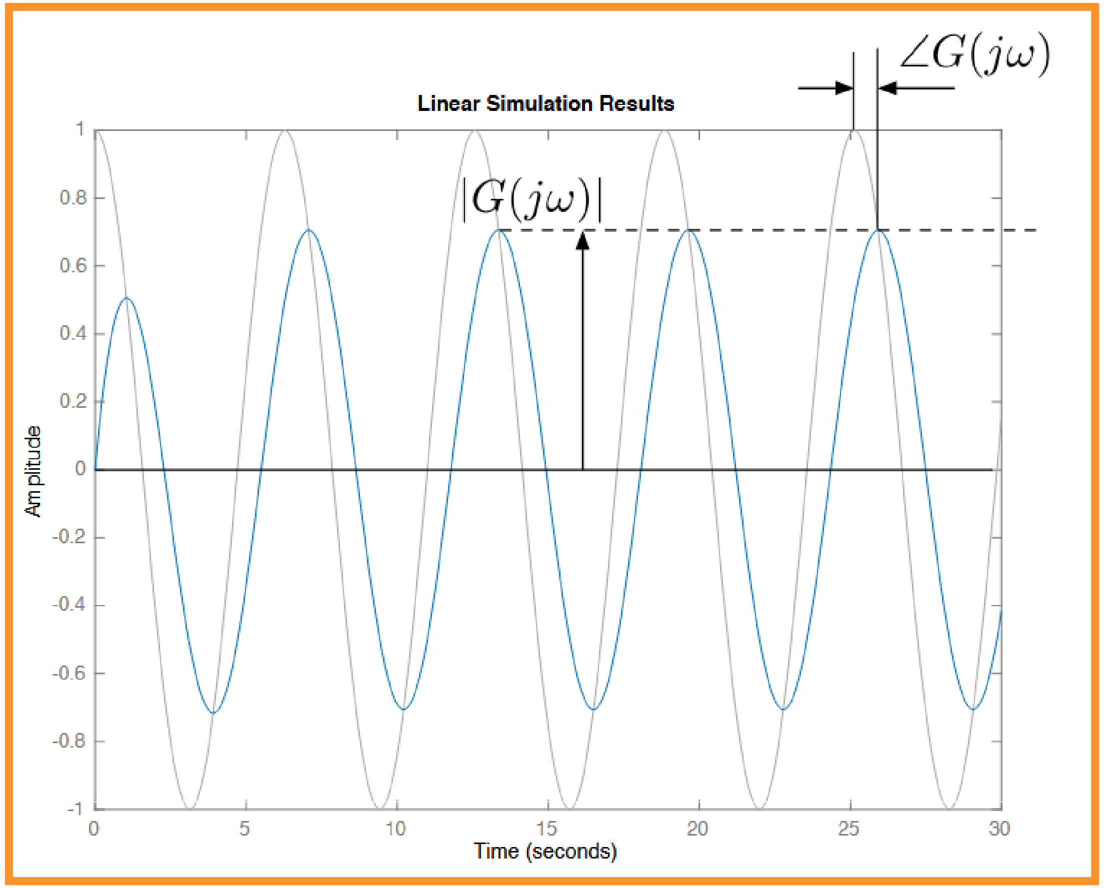
$$M = |G(j\omega)| \quad \text{and} \quad \varphi = \angle(G(j\omega))$$

So now our response becomes: $$y_{ss}(t) = \frac{1}{2}\underbrace{Me^{j\varphi}}_{G(j\omega)}e^{j\omega t} + \frac{1}{2}\underbrace{Me^{-j\varphi}}_{G(-j\omega)}e^{-j\omega t}$$

Reformulating: $$y_{ss}(t) = M\cos(\omega t + \varphi)$$

We can conclude that for a sinusoidal input, the output is just another sinusoid with the **same frequency, shifted by a phase φ and multiplied by the magnitue M.**

Motivation



$\angle G(j\omega)$
Linear Simulation Results
$|G(j\omega)|$
Amplitude
Time (seconds)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30

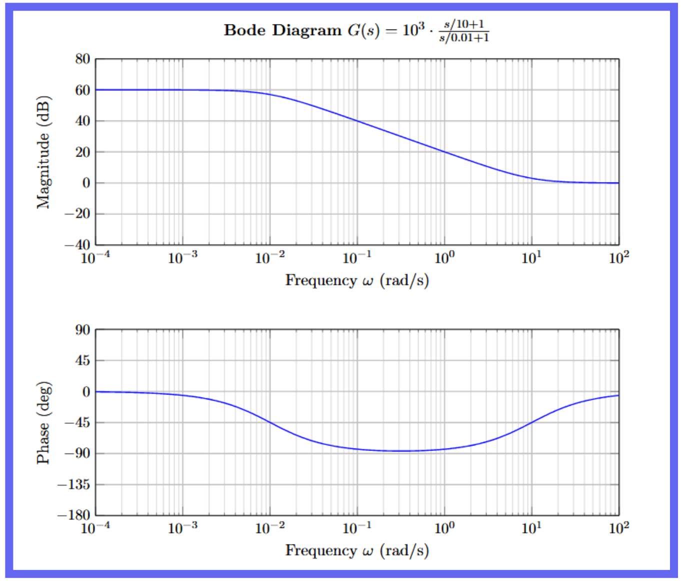
Motivation

$$y_{ss} = |G(j\omega)| \cos(\omega t + \angle G(j\omega))$$

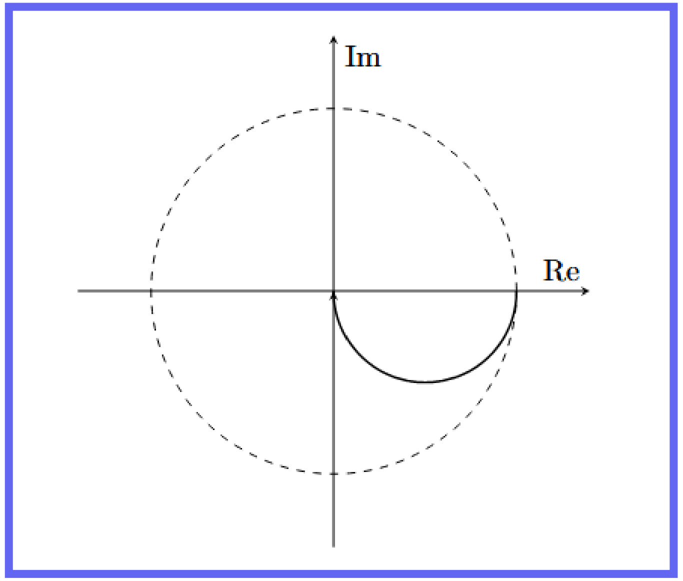
We can see, that magnitude and phase of our TF **do not depend on time t** but only on the **frequency ω**.

Therefore it would be nice to have a plot showing how our frequency response (output) behaves when changing the input frequency. For that we will explore 2 options:

1. **Bode Plot**: The magnitude and the phase of the TF are in 2 seperate plots



2. **Polar / Nyquist Plot**: A parametric curve of the TF with ω implicit



Bode Plot

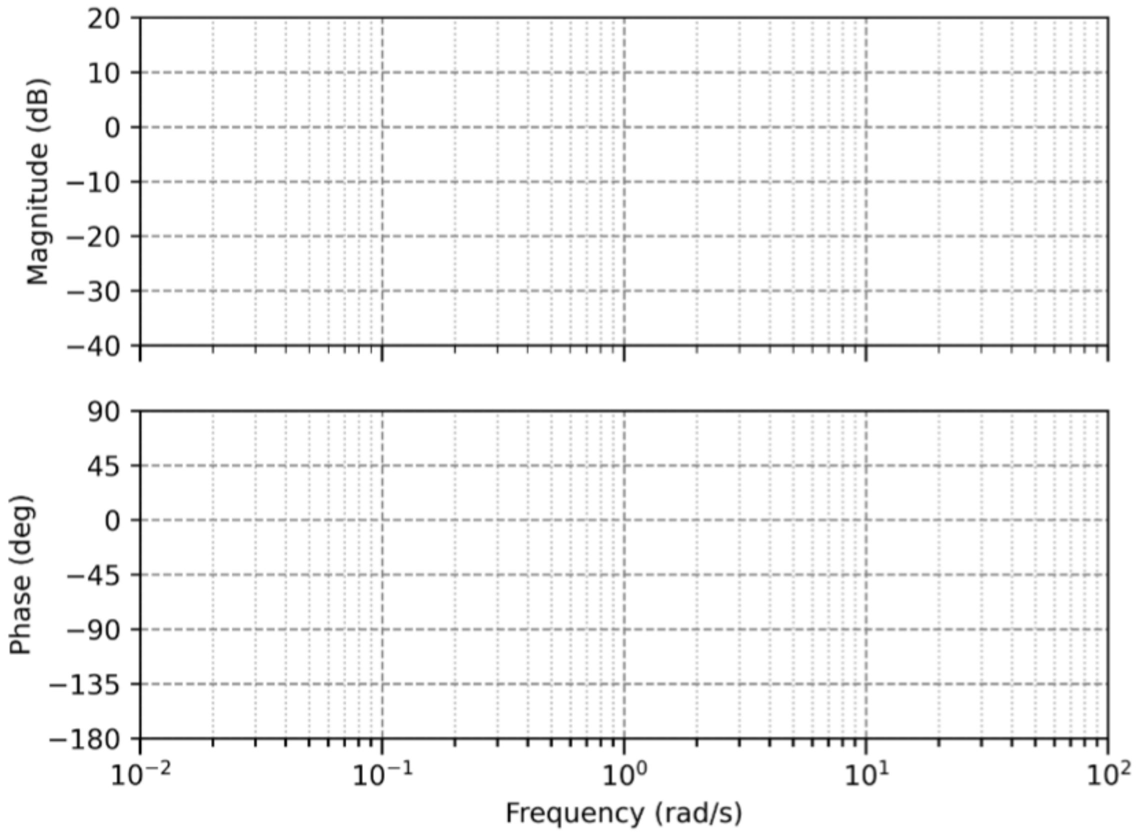
Bode Plot

As already mentioned, we will have 2 «subplots». One for the magnitude and one for the phase of our TF. Together, they are the **Bode Plot**.
Both of them have a **log scale** of the **frequency** on the horizontal axis!

Magnitude Plot: Has the magnitude in dB on the vertical axis, where

$$|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|$$

Phase Plot: Has the phase in deg on the vertical axis



Magnitude in dB (deciBel)

The good thing is, that now for both of the vertical axes:

$$\log(a \cdot b) = \log(a) + \log(b)$$
$$\angle(a \cdot b) = \angle(a) + \angle(b)$$

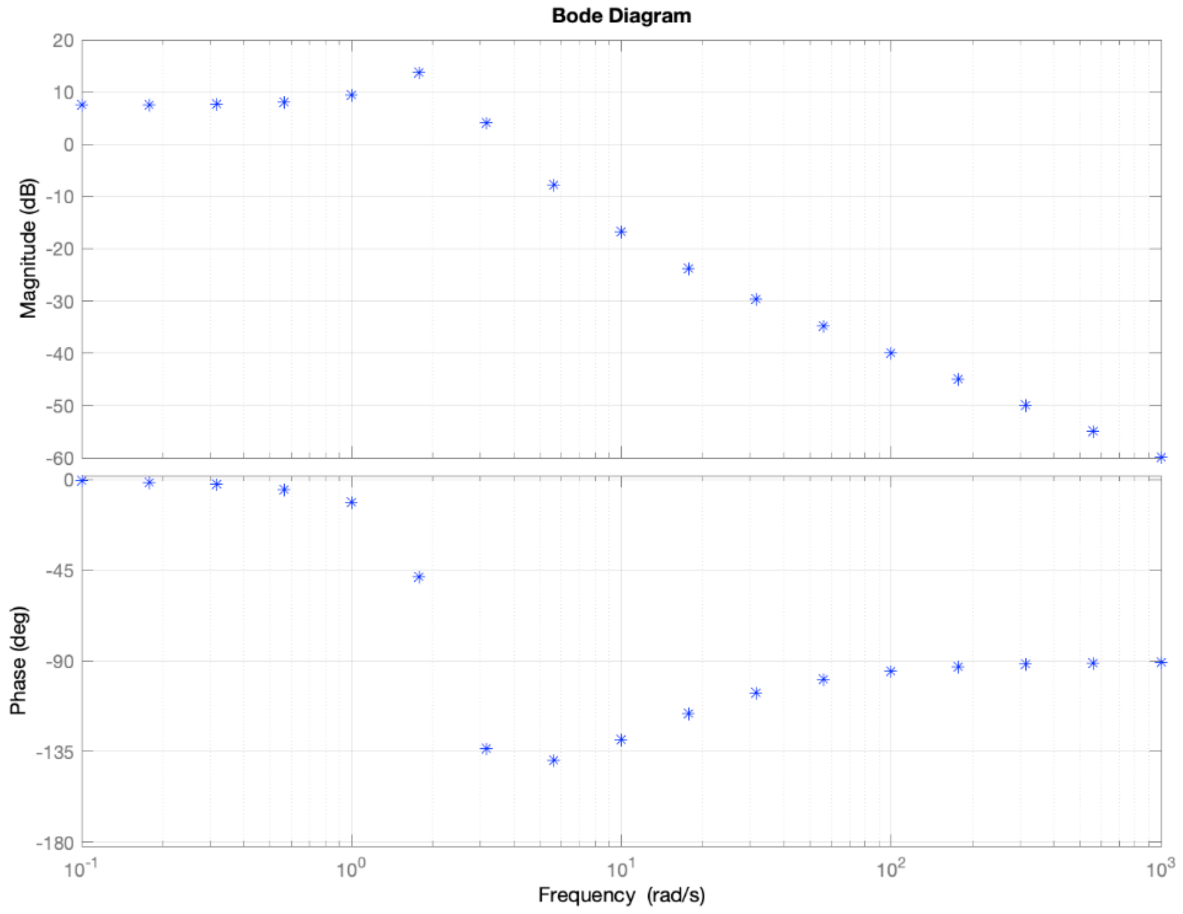
Which means that we can just multiply 2 random TF, but their Bode Plots just get added up.
This is going to be super useful in the next steps!

Decimal scale	[dB] scale
100	40
10	20
3.16	10
2	6.02
1	0
$1/\sqrt{2}$	-3.01
0.1	-20
0.01	-40
0	$-\infty$
$\lvert G(j\omega)\rvert$	$20 \cdot \log_{10} \lvert G(j\omega)\rvert$
$10^{\lvert G(j\omega)\rvert_{[dB]}/20}$	$\lvert G(j\omega)\rvert_{[dB]}$

Bode Plot

Okay, but lets get back to what the bode plot really shows us:

Let us try out some values for ω and see how the magnitude and phase change when having a sinusoidal input $u(t) = sin(\omega t)$.
You can see this on the right:



We now want to **derive a model with rules** that helps sketch the plot without having to calculate the magnitude and phase for every value of ω …

(similar idea as the root locus for different k)

Bode Plot Decomposition

Bode Form

$$G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\cdots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\cdots\left(\frac{s}{-p_{n-q}}+1\right)}$$

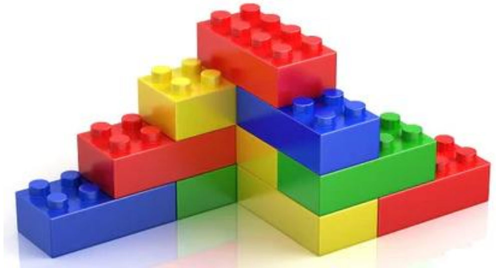
Remember our previously introtoced form of a TF?

Just as in the root locus, we can see everything is factorized.

As already mentioned, we can multiply TF but their bode plots just get added up.
Meaning: We can just imagine our G(s) as a multiplied version of simpler TF.

G(s) consists of:

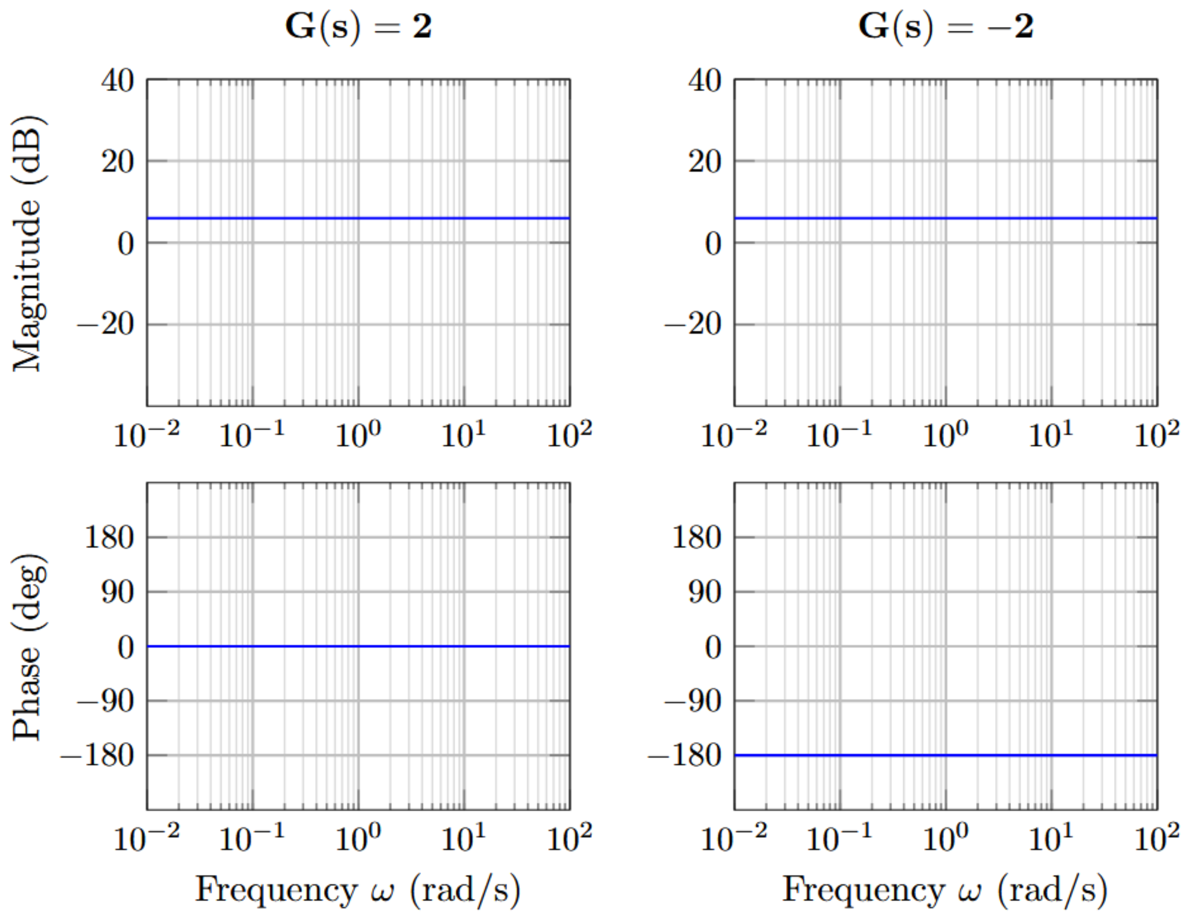
- Gain k
- Integrator / Differentiator
- Poles
- Zero



In the end we can just superposition the bode plots of these simpler bits!

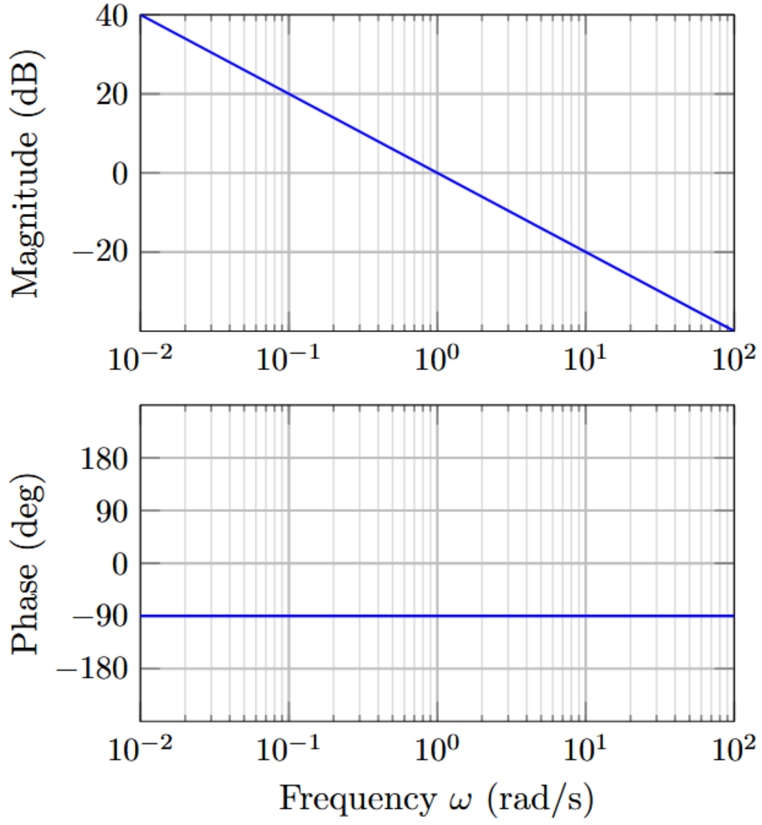
Basic Elements: Constant Gain

$$G(s) = k$$



Basic Elements: Integrator and Differentiator

$$G(s) = \frac{1}{s} \qquad G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega}$$

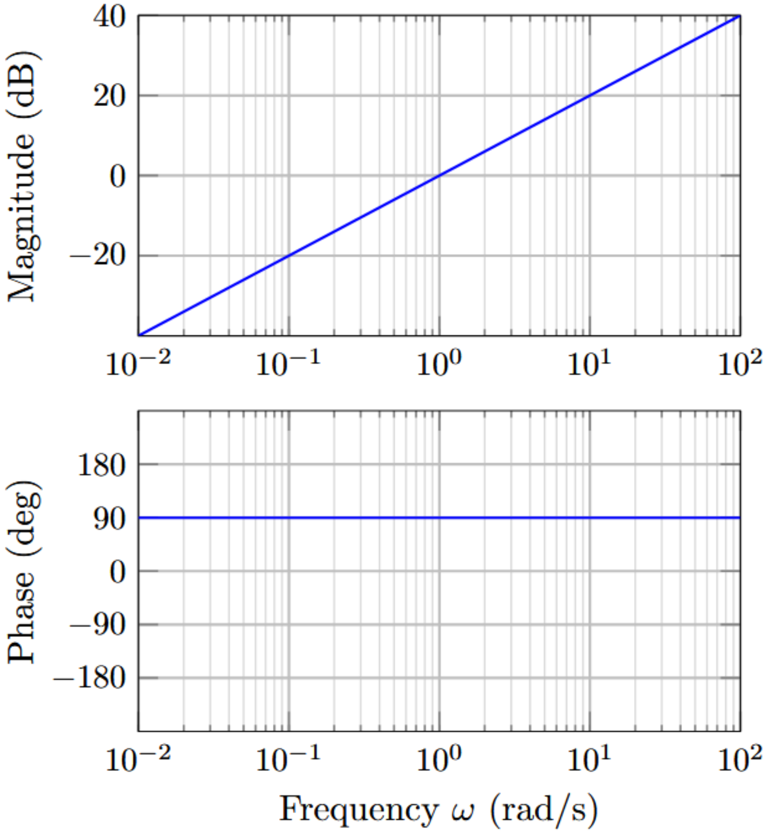


Integrator has a slope of -20dB/decade

Both have 0dB at $\omega = 1$

Differentiator has a slope of 20dB/decade

$$G(s) = s \qquad G(j\omega) = j\omega$$



Basic Elements: Pole

$$G(s) = \frac{1}{\frac{s}{-p} + 1}$$

Let's derive this behaviour a little bit and see how the plots on the right get constructed:

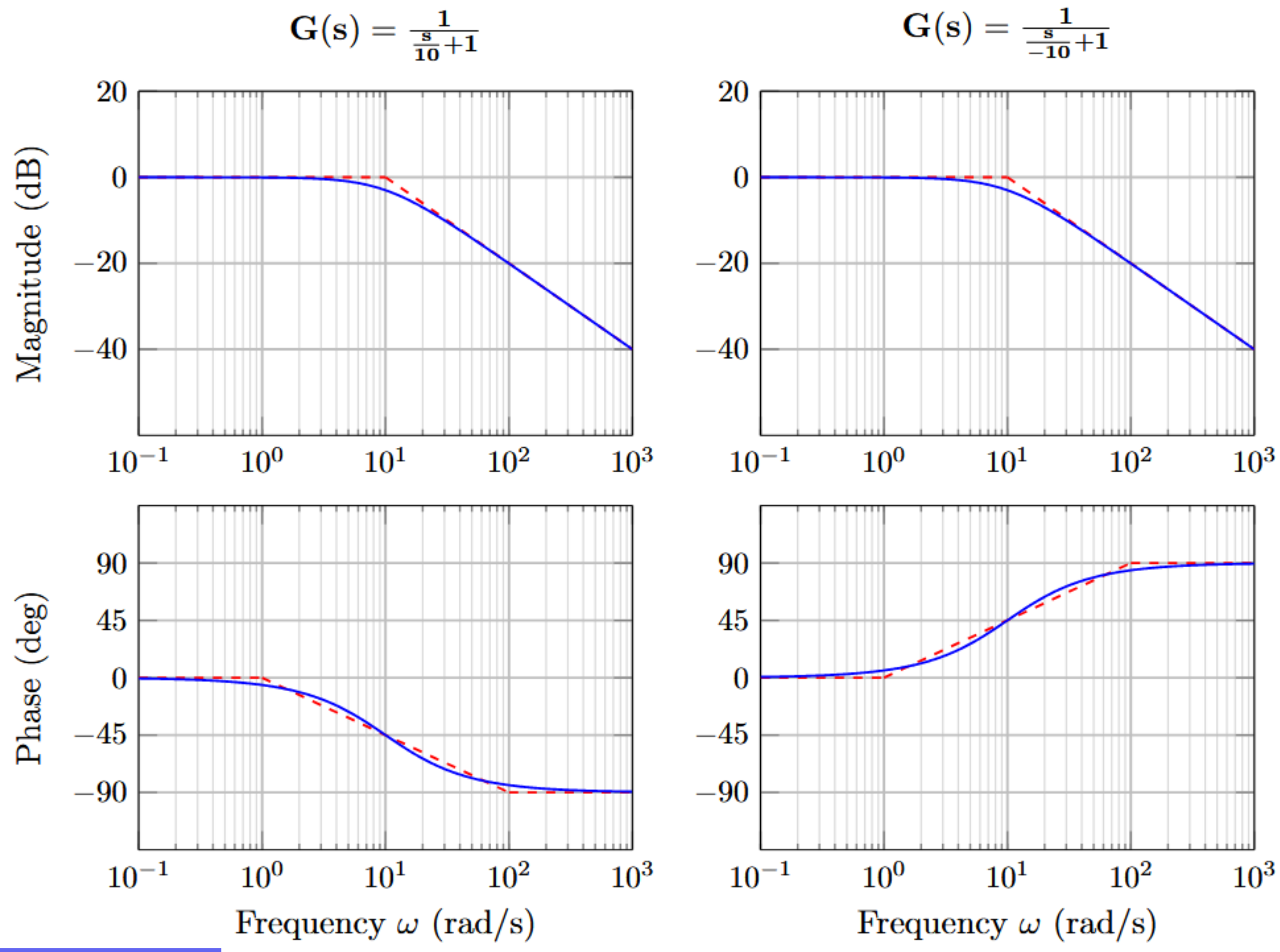
$$\lim_{\omega \to 0} G(j\omega) = \frac{1}{\frac{j\omega}{-p}+1} = \frac{1}{1} \quad \Rightarrow |\ldots| = 0\,\mathrm{dB} \quad \Rightarrow \angle(\ldots) = 0^\circ$$

$$\lim_{\omega \to \infty} G(j\omega) = \frac{1}{\frac{j\omega}{-p}+1} \approx \frac{-p}{j\omega} = j\frac{p}{\omega}$$

$$\Rightarrow |\ldots| \approx \frac{|p|}{\omega} = [20\log_{10}(|p|) - 20\log_{10}(\omega)]\ \mathrm{dB}$$

$$\Rightarrow \angle(\ldots) = -90^\circ \ \text{(for stable pole)}$$

$$\omega = |p| \quad G(j\omega) = \frac{1}{j+1} \quad \Rightarrow |\ldots| = \frac{1}{\sqrt{2}} \approx -3\,\mathrm{dB} \quad \Rightarrow \angle(\ldots) = -45^\circ \ \text{(for stable pole)}$$



1. Takeaways

- **The magnitude only starts changing at the position of the respective pole.**
  It changes with -20dB / decade.

- **The phase starts changing one decade before the pole and ends one decade after. So from 0.1|p| until 10|p|**
  If it's a stable pole, it changes -90 degress, for unstable poles, +90 degrees

Basic Elements: Zero

$$G(s) = \frac{s}{-z} + 1$$

Bascially the same derivation as for the pole, but some things are inverted

Minimum Phase Zero:

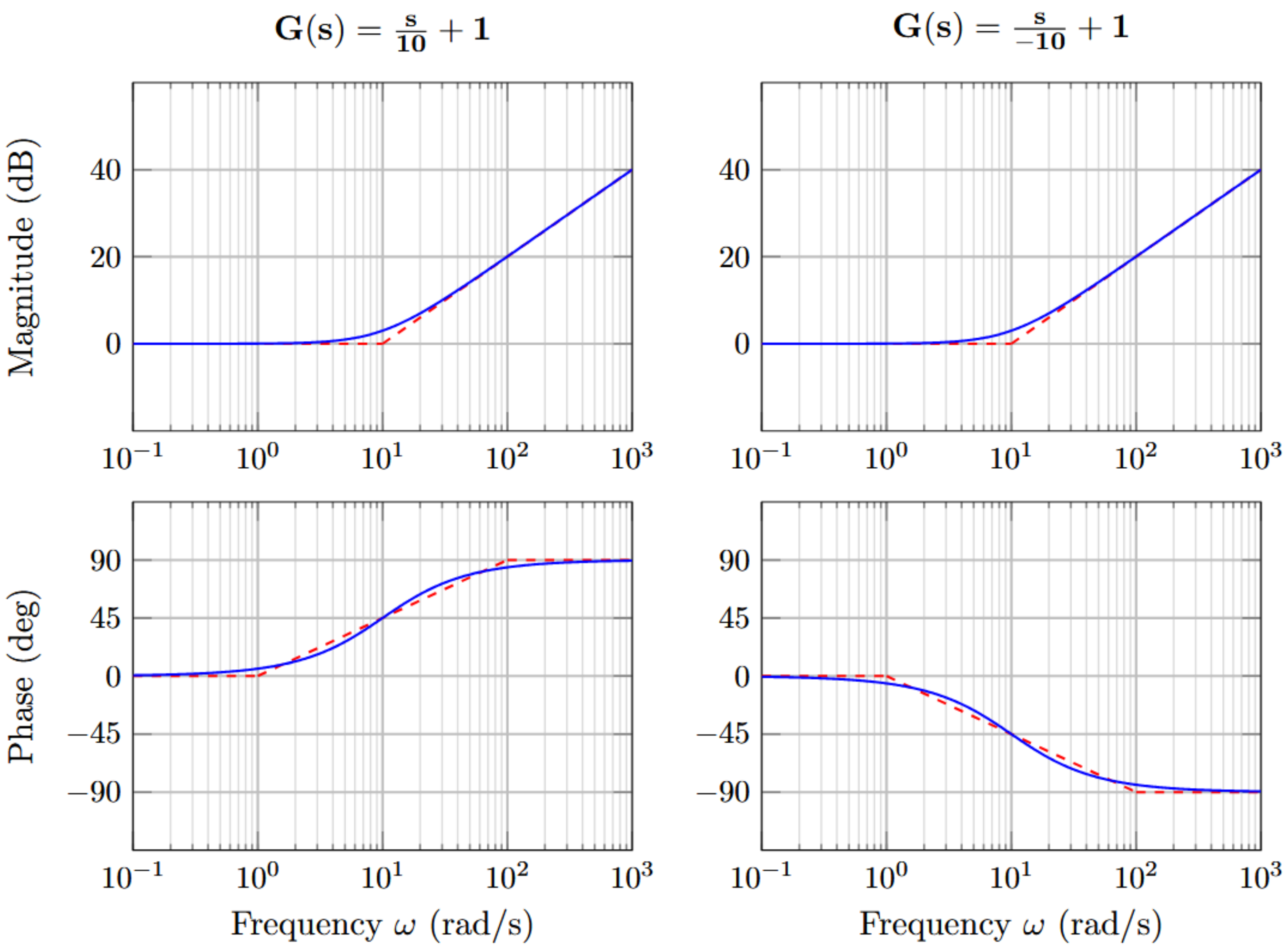
+20dB / decade

+90 degrees

Non Minimum Phase Zero:

+20dB / decade

-90 degrees

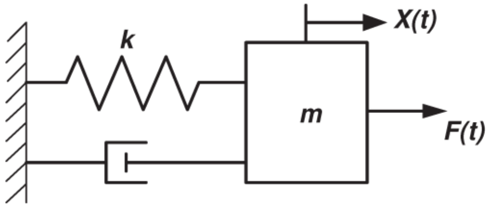


Basic Elements: Complex Conjugate Stable Poles

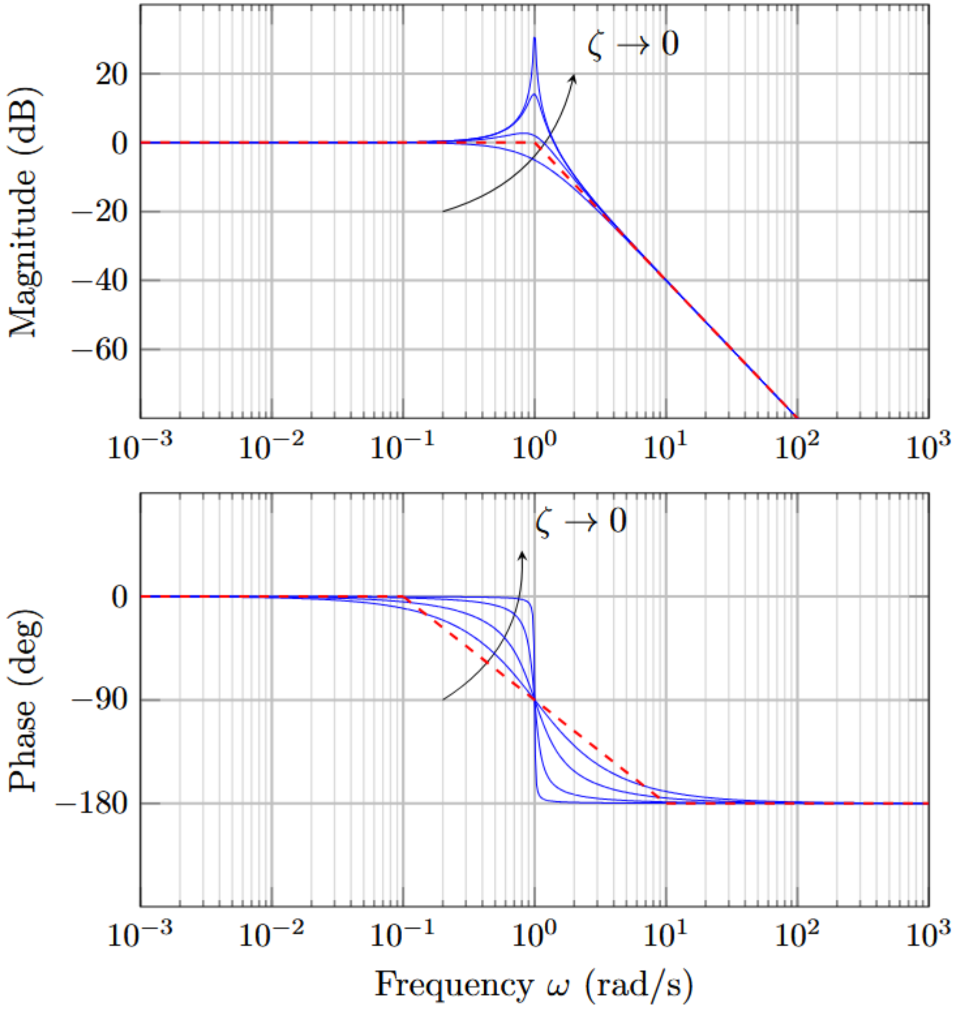
Basically just two poles at the same frequency, so just double everything that would happen for one pole.

BUT see how at the frequency the position of the pole has a peak, and how the phase changes more instantaneously, if the damping ratio is decreased.

$$G(s) = \frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



This TF comes from a system like on the left, where you not only have a spring but also some damping



Derivation Complex Conjugate Stable Poles

For you to look at if you want. Won't go through it now. However it is nice!

$$G(s) = \frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega \to 0, \quad G(j\omega) \approx 1 \qquad |G(j\omega)| \approx 1 = 0\,\text{dB}, \qquad \angle G(j\omega) \approx 0^\circ$$

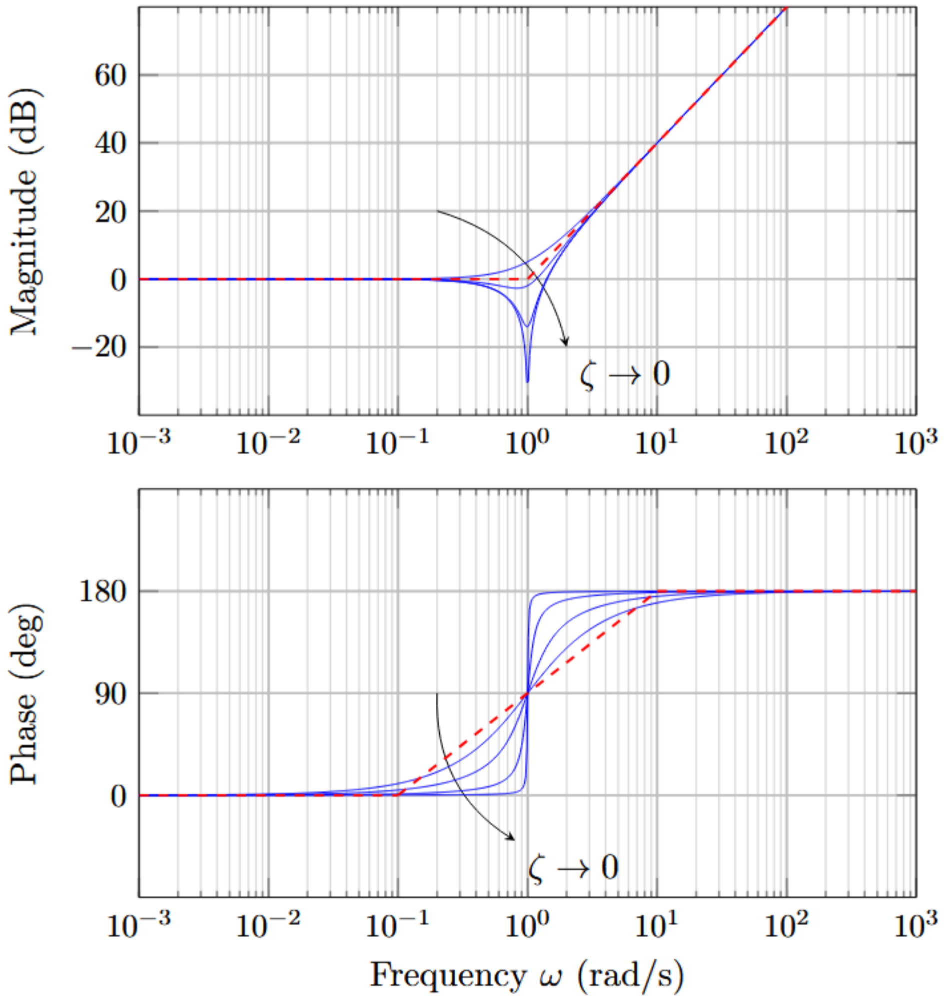
$$\omega \to \infty, \quad G(j\omega) \approx -\frac{\omega_n^2}{\omega^2} \qquad |G(j\omega)| \approx \frac{\omega_n^2}{\omega^2} = [40\ln(\omega_n) - 40\ln(\omega)]\ \text{dB}, \qquad \angle G(j\omega) \approx -180^\circ$$

$$\omega = \omega_n, \quad G(j\omega) = \frac{1}{2\zeta j} = -\frac{j}{2\zeta} \qquad |G(j\omega)| = \frac{1}{2\zeta}, \qquad \angle G(j\omega) = -90^\circ$$

Basic Elements: Complex Conjugate Minimum - Phase Zero

$$G(s) = s^2/\omega_n^2 + 2\zeta s/\omega_n + 1$$

Same as for the poles but inverted again



Super Ultimate Slide

Bode Plot Rules

1. Write the **TF in Bode form!!**
2. Plot magnitude and phase of **Bode gain**
3. Do not forget to give **magnitude in dB**
4. Superposition all basic compontents. Remember:

   - The **magnitude change** starts **at the position** (frequency) of respective pole or zero
   - The **phase change starts one decade to the left and ends one decade to the right** of the position
   - Integrators and Differentiators have magnitude [dB] = 0 at $\omega = 1$
   - **For complex conjugate** poles remember the **peak and sudden phase change**

1. We are allowed to draw straight lines. Keep in mind however, that this is an approximation.

$$G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\cdots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\cdots\left(\frac{s}{-p_{n-q}}+1\right)}$$

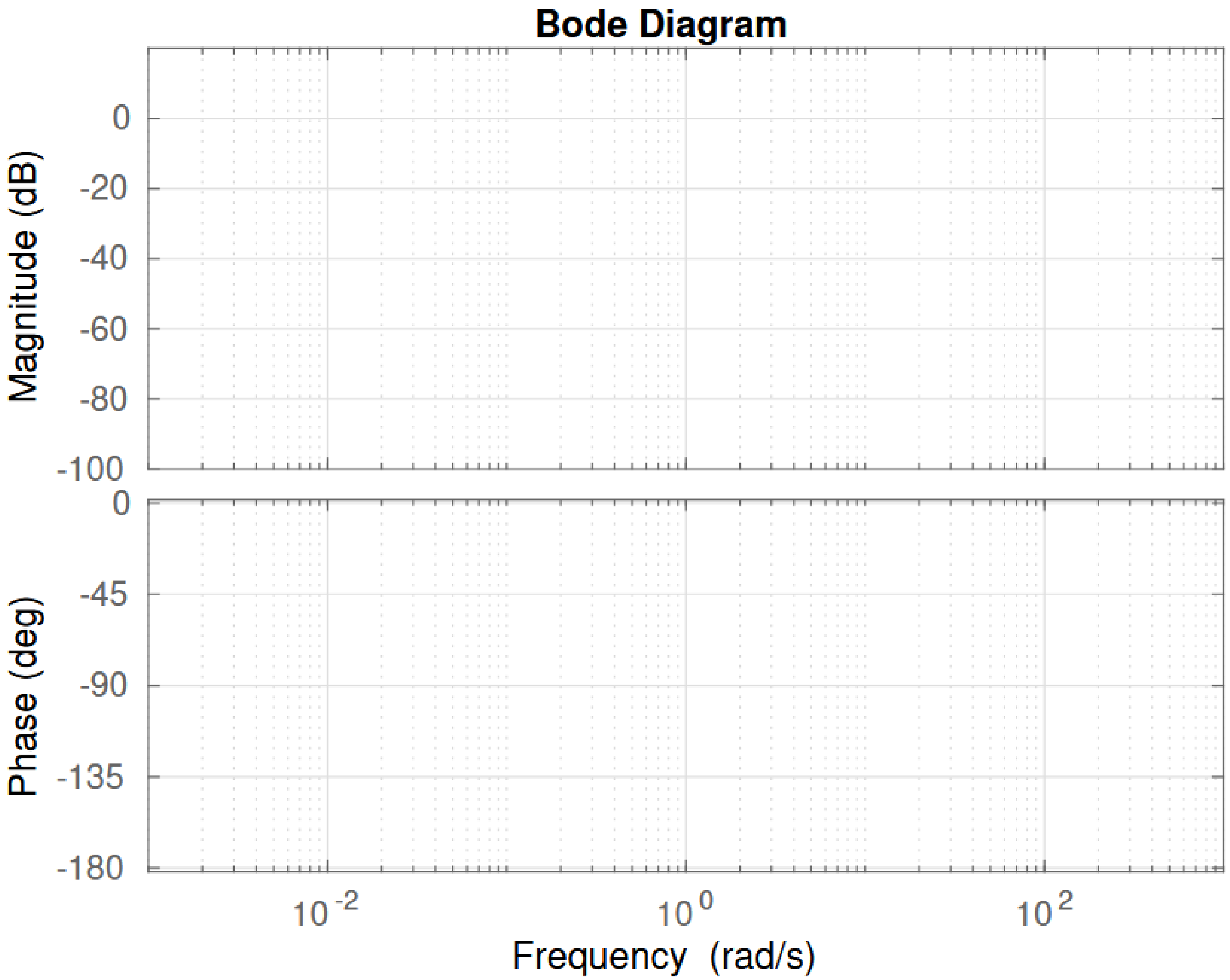
$$|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|$$

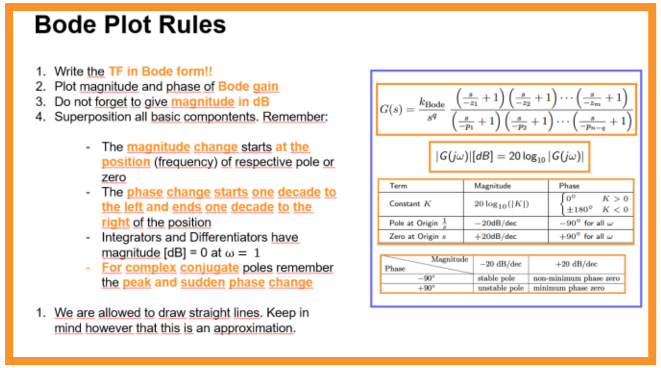
Term	Magnitude	Phase
Constant K	$20\log_{10}(\lvert K\rvert)$	$\begin{cases} 0^\circ & K>0 \\ \pm180^\circ & K<0 \end{cases}$
Pole at Origin $\frac{1}{s}$	-20dB/dec	-90° for all ω
Zero at Origin s	$+20$dB/dec	$+90^\circ$ for all ω

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
-90°	stable pole	non-minimum phase zero
$+90^\circ$	unstable pole	minimum phase zero

Example

$$G_1(s) = \frac{100s}{(10s+1)(s+10)}$$





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   - For complex conjugate poles remember the peak and sudden phase change

1. We are allowed to draw straight lines. Keep in mind however that this is an approximation.

$$G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\cdots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\cdots\left(\frac{s}{-p_{n-q}}+1\right)}$$

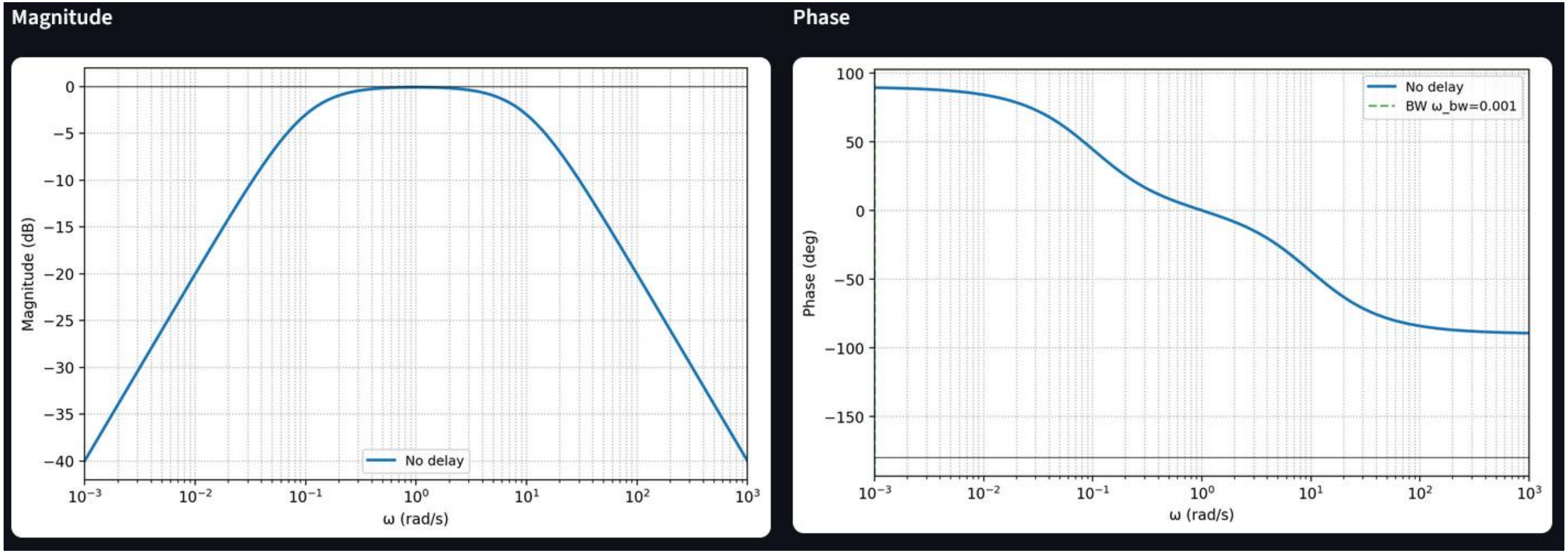
$$|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|$$

Term	Magnitude	Phase
Constant K	$20\log_{10}(\lvert K\rvert)$	$\begin{cases} 0^\circ & K>0 \\ \pm180^\circ & K<0 \end{cases}$
Pole at Origin $\frac{1}{s}$	-20dB/dec	-90° for all ω
Zero at Origin s	$+20$dB/dec	$+90^\circ$ for all ω

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
-90°	stable pole	non-minimum phase zero
$+90^\circ$	unstable pole	minimum phase zero

Example

$$G(s) = 10 \cdot \frac{s}{(\frac{s}{0.1}+1)(\frac{s}{10}+1)}$$



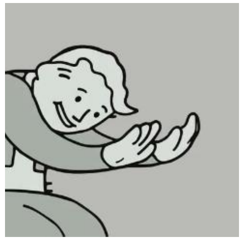
How did I plot the solution so easily? With the cool **Bode-Plot-Tool** that I built for the course.
Not published yet of the course, but you can have a look here: https://bodetooltest.streamlit.app/

Inverting a Transfer Function

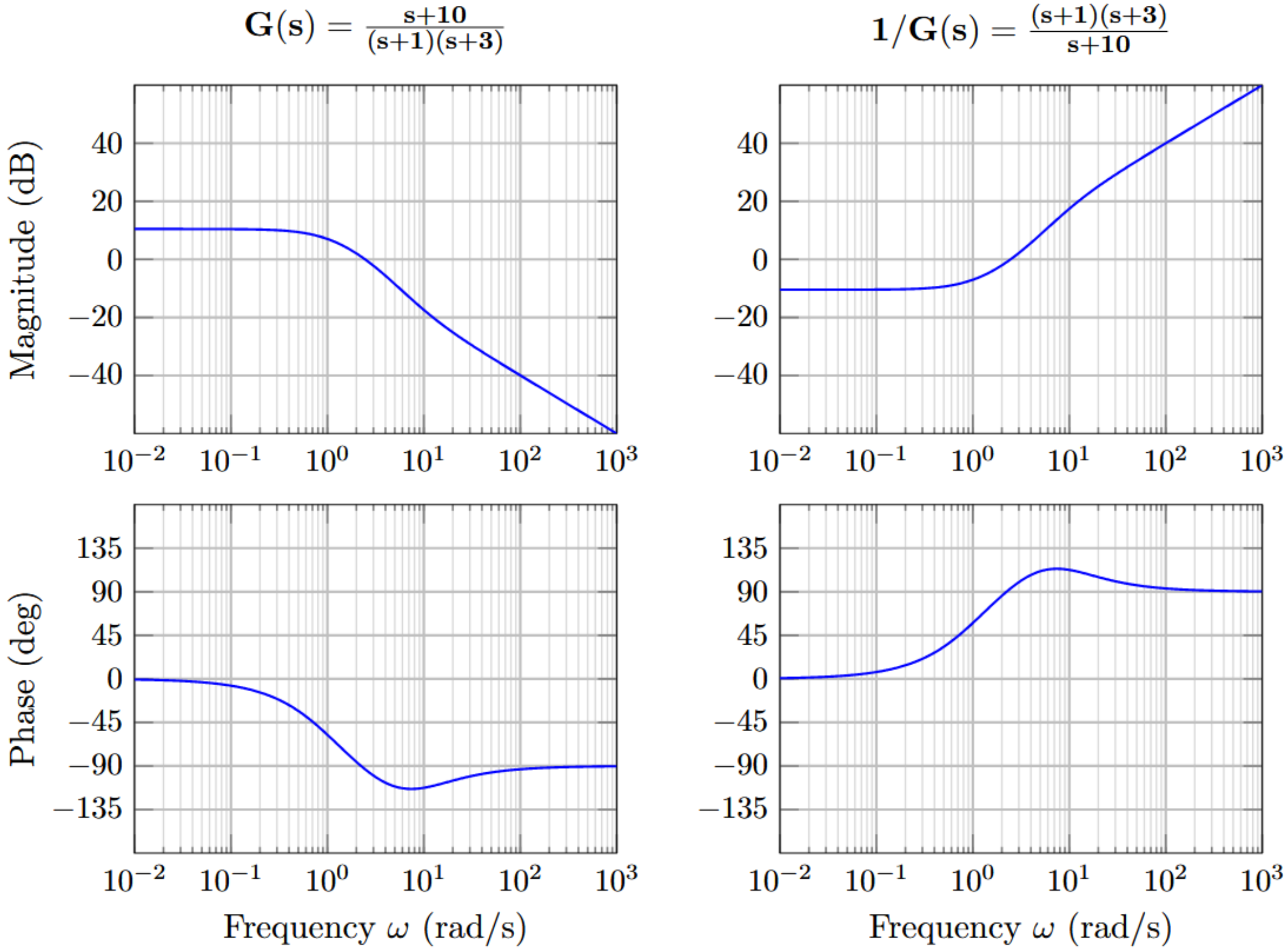
Basically we just mirror the Bode plot on the horizontal axis.

Makes sense, since we can write

$$|1|[\mathrm{dB}] - |G(s)|[\mathrm{dB}] = -|G(s)|[\mathrm{dB}]$$
$$\angle(1) - \angle(G(s)) = -\angle(G(s))$$



A zero ist just an inverted pole??!!



Frequency Response Idea

Let’s step back again and think about why we do this:

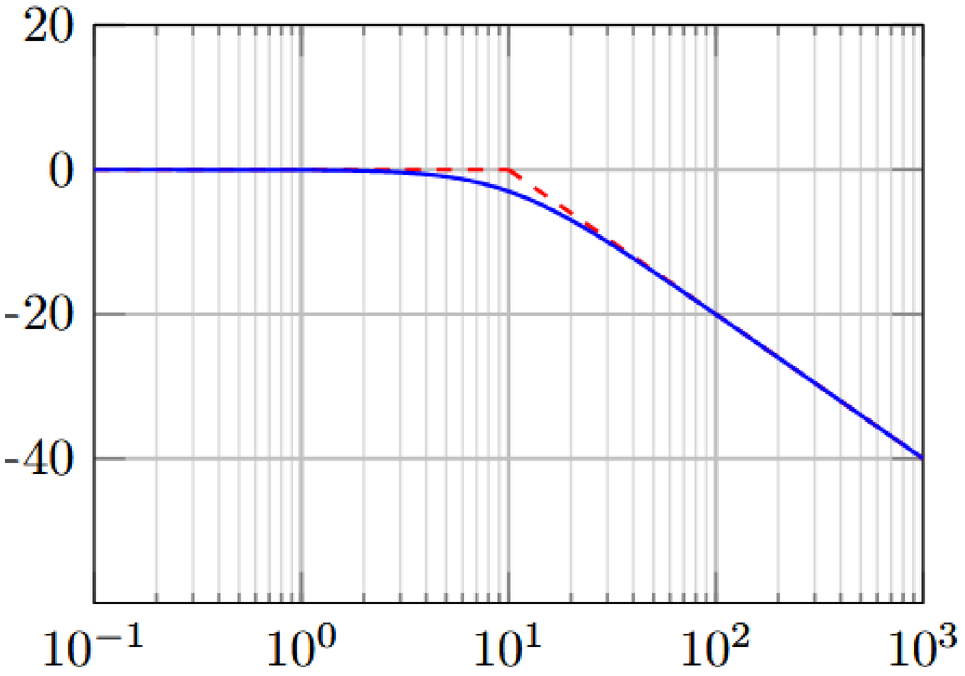
We would like to know how the systems responds to a **sinusoidal input with a certain frequency.**

What for example, you excite a system at a certain frequency, that is way larger than the frequency the pole is located on?

You can see that the magnitude in dB will go to -∞ or just to 0 for non dB. Meaning that our systems response will just be 0 for to high frequencies.

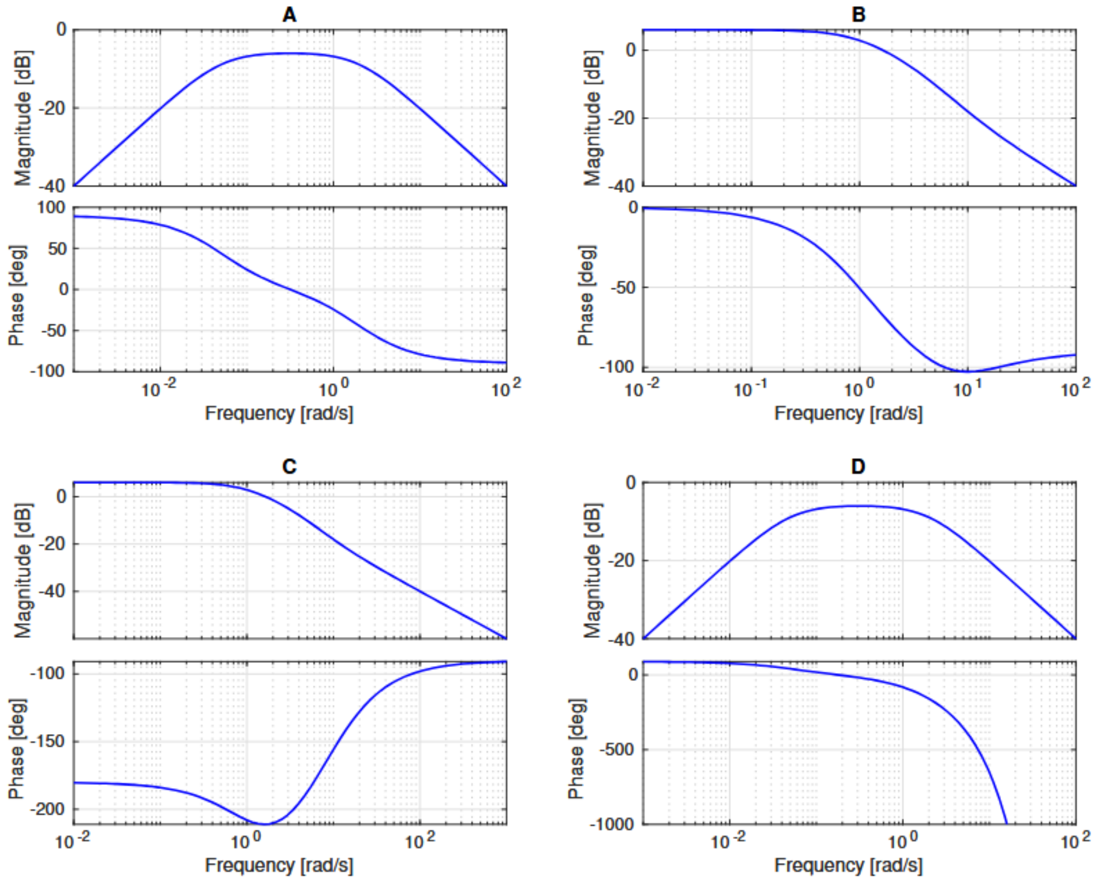
We can use the bode plot to analyse this behaviour and tune how our system should react to certain frequencies

$$G(s) = \frac{1}{\frac{s}{10}+1}$$



FS 2024

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
$-90°$	stable pole	non-minimum phase zero
$+90°$	unstable pole	minimum phase zero



Match the TF to their bode plots

$$G_1(s) = \frac{s}{s^2 + 2s + 0.1} \qquad G_2(s) = e^{-s}\frac{s}{s^2 + 2s + 0.1}$$

$$G_3(s) = \frac{s + 10}{(s+1)(s+5)} \qquad G_4(s) = \frac{s + 10}{(s+1)(s-5)}$$

A) (A, G_1), (B, G_3), (C, G_4), (D, G_2)

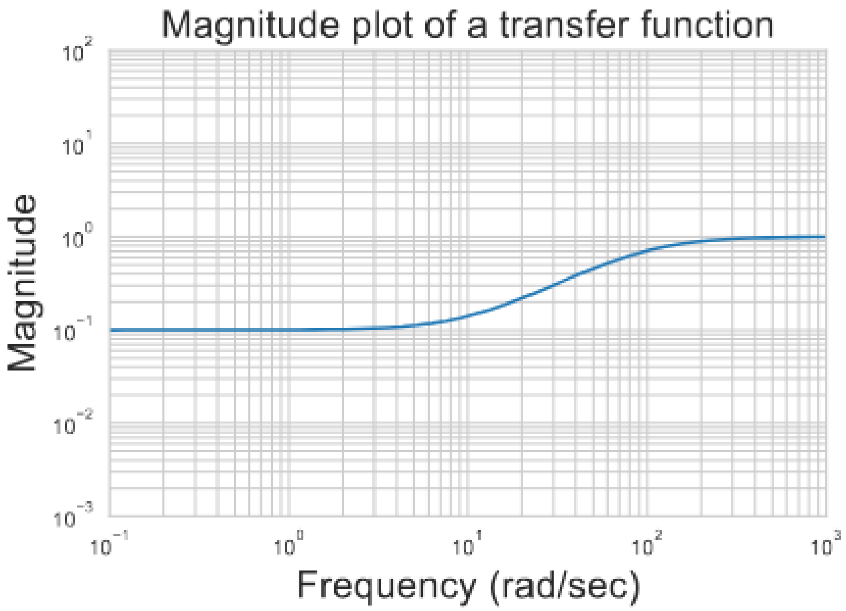
C) (A, G_2), (B, G_3), (C, G_4), (D, G_1)

B) (A, G_1), (B, G_4), (C, G_3), (D, G_2)

D) (A, G_3), (B, G_1), (C, G_2), (D, G_4)

FS 2018

Phase \ Magnitude	−20 dB/dec	+20 dB/dec
−90°	stable pole	non-minimum phase zero
+90°	unstable pole	minimum phase zero



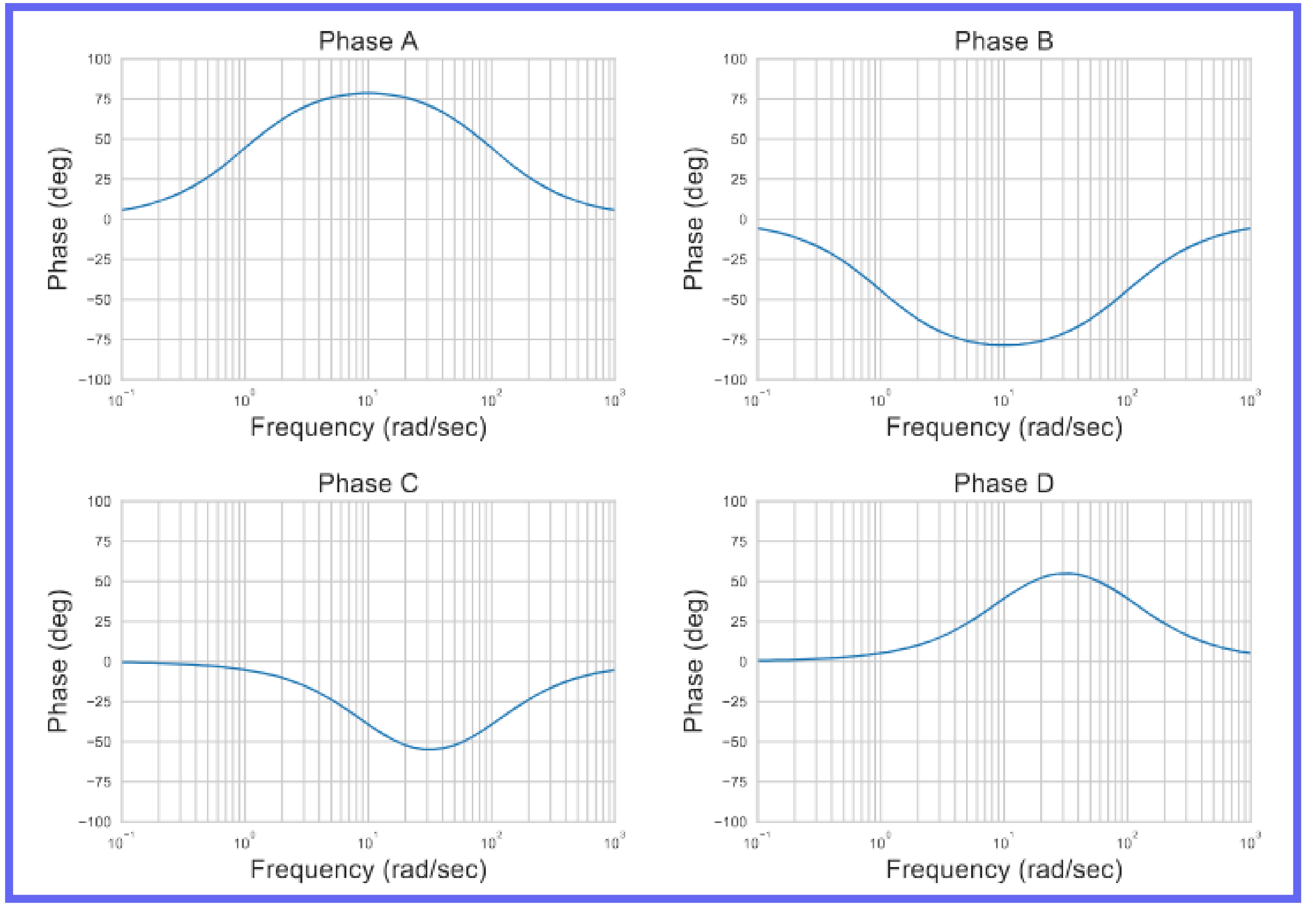
Which of the following phase plots corresponds to the given magnitude plot?

A)

C)

B)

D)



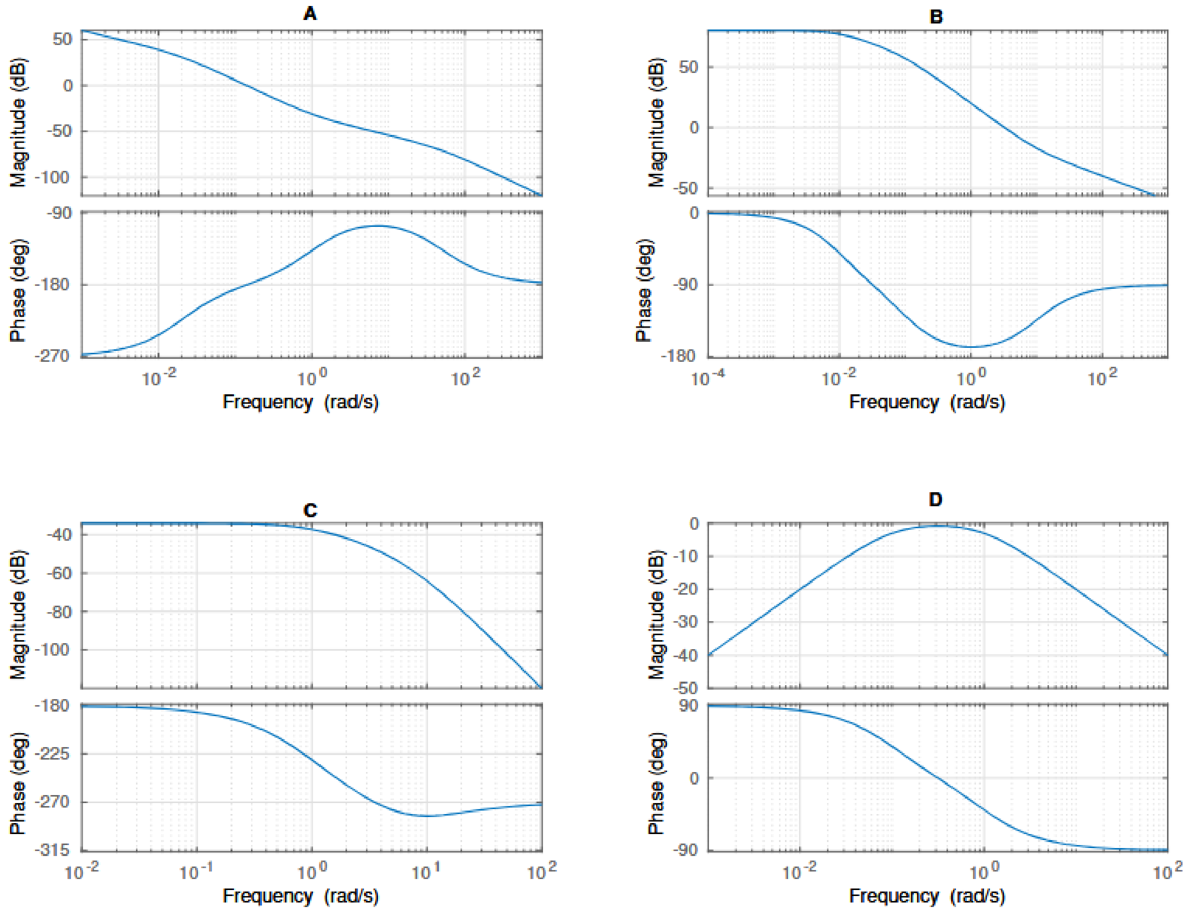
Q&A Session / Done

Next Time: Polar Plot and Nyquist

Feedback

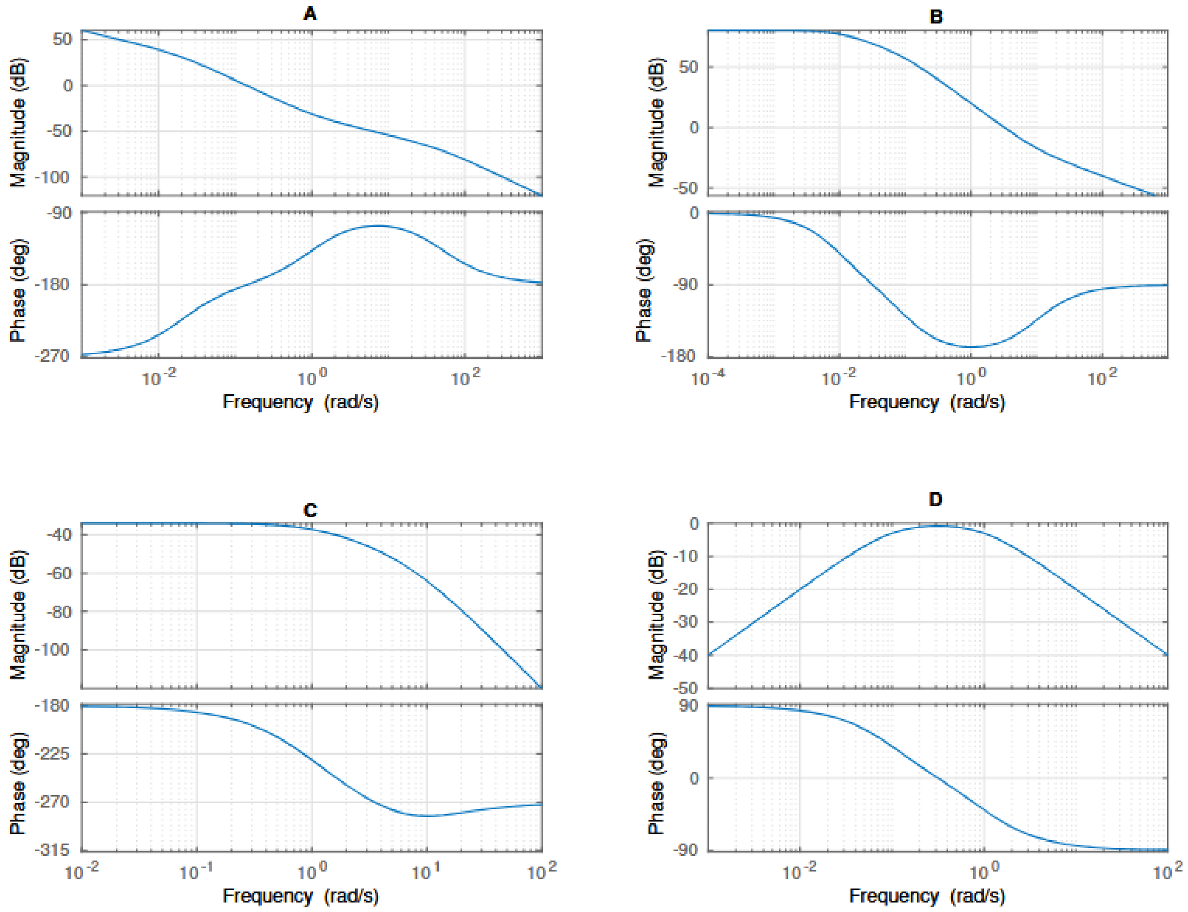
jschultev.github.io/personal_website/Feedback

HS 2022



Transfer Function	A	B	C	D
$L_1(s) = \dfrac{s+1}{s(s+50)(s-0.02)}$				
$L_2(s) = \dfrac{s}{(s+1)(s+0.01)}$				
$L_3(s) = \dfrac{s+10}{(s+0.01)(s+0.1)}$				
$L_4(s) = \dfrac{1}{(s+1)(s-10)(s+5)}$				

HS 2022



Transfer Function	A	B	C	D
$L_1(s) = \dfrac{s+1}{s(s+50)(s-0.02)}$	x			
$L_2(s) = \dfrac{s}{(s+1)(s+0.01)}$				x
$L_3(s) = \dfrac{s+10}{(s+0.01)(s+0.1)}$		x		
$L_4(s) = \dfrac{1}{(s+1)(s-10)(s+5)}$			x	